

THE CLASSIFICATION OF p -NILPOTENT RESTRICTED LIE ALGEBRAS OF DIMENSION AT MOST 4

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ABSTRACT. In this paper we obtain the classification of p -nilpotent restricted Lie algebras of dimension at most four over a perfect field of characteristic p .

1. INTRODUCTION

In this paper we initiate the classification of small dimensional restricted Lie algebras. Similar classifications for ordinary Lie algebras have a long history. The classification of all nilpotent Lie algebras up to dimension five over any field has been known for a long time. However, in dimension 6, the characterization depends on the underlying field. In 1958 Morozov [6] gave a classification of nilpotent Lie algebras of dimension 6 over a field of characteristic zero, see also [3, 7, 5] for a classification over other fields. These classifications, however, differ and it was not easy to compare them until recently that de Graaf [2] gave a complete classification over any field of characteristic other than 2. de Graaf's approach can be verified computationally and was later revised and extended to characteristic 2 in [1]. The classification in dimensions more than 6 is still in progress, see for example [9, 10].

In this paper we give a list of $[p]$ -nilpotent restricted Lie algebras of dimension at most 4 over a perfect field of characteristic $p \geq 3$. Let L be a Lie algebra over a field \mathbb{F} of characteristic p . Recall that L is said to be *restrictable* if L affords a $[p]$ -map $x \mapsto x^{[p]}$ that satisfies the following properties:

- (1) $(\text{ad } a)^p = \text{ad } a^{[p]}$;
- (2) $(\alpha a)^{[p]} = \alpha^p a^{[p]}$;
- (3) $(a + b)^{[p]} = a^{[p]} + b^{[p]} + \sum_{i=1}^{p-1} s_i(a, b)$

where $s_i(a, b)$ is given by the formula

$$(\text{ad } (a \otimes X + b \otimes 1))^{p-1} (a \otimes 1) = \sum_{i=1}^{p-1} i s_i(a, b) \otimes X^{i-1}$$

interpreted in the Lie algebra $L \otimes \mathbb{F}[X]$ over the polynomial ring $\mathbb{F}[X]$. A Lie algebra L with a given $[p]$ -map $x \mapsto x^{[p]}$ is said to be *restricted*.

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Recall that a restricted Lie algebra L is called $[p]$ -nilpotent if there exists an integer n such that $L^{[p]^n} = 0$. Let L be a finite-dimensional $[p]$ -nilpotent restricted Lie algebra. Then, by Engel's Theorem, L is nilpotent. Note that $(x + y)^{[p]} = x^{[p]} + y^{[p]}$ modulo $\gamma_p(L)$, for every $x, y \in L$. Here, $\gamma_i(L)$ is the i -th term of the lower central series of L . So if the nilpotency class of L is smaller than p , then the $[p]$ -map is a semilinear transformation from L to the center $Z(L)$ of L . Let $\varphi_1, \varphi_2 : L \rightarrow Z(L)$ be two semilinear transformations. Then the restricted Lie algebras (L, φ_1) and (L, φ_2) are isomorphic if and only if there exists $A \in \text{Aut}(L)$ such that

$$x\varphi_1 A = xA\varphi_2 \quad \text{holds for all } x \in L.$$

Hence, φ_1 and φ_2 define isomorphic restricted Lie algebras if and only if there exists $A \in \text{Aut}(L)$ such that $A\varphi_1 A^{-1} = \varphi_2$; that is, they are conjugate under the automorphism group of L . In this case we say that the $[p]$ -maps φ_1 and φ_2 are *equivalent*. This defines a left action of $\text{Aut}(L)$ on the set of $[p]$ -maps and the isomorphism classes of restricted Lie algebras correspond to the $\text{Aut}(L)$ -orbits under this action.

Our work is motivated by the isomorphism problem for enveloping algebras of restricted Lie algebras. We are interested in understanding when two non-isomorphic restricted Lie algebras can have isomorphic restricted enveloping algebras and it makes sense to examine the class of $[p]$ -nilpotent restricted Lie algebras. The first step towards this is a classification of such restricted Lie algebras in small dimensions.

2. THE MAIN RESULT

The main theorem of the paper is a classification of $[p]$ -nilpotent restricted Lie algebra with dimension at most 4 over perfect fields \mathbb{F} . We use as our starting point, the classification of nilpotent Lie algebras of dimension 4 and classify the possible equivalence classes of $[p]$ -maps on these Lie algebras.

In dimensions 1 and 2, there is a unique isomorphism type of nilpotent Lie algebras. There are two nilpotent Lie algebras of dimension 3, one is abelian and the other one is nilpotent of class 2. There are three nilpotent Lie algebras of dimension 4, one is abelian, one nilpotent of class 2, and one nilpotent of class 3. If p is greater than the nilpotency class, then a $[p]$ -map is semilinear, this is however not always the case in characteristic 3 and 5.

Our main result assumes that the field is perfect. The reason for this is that we often need that the Frobenius automorphism $x \mapsto x^p$ of \mathbb{F} is surjective. If $\text{char } \mathbb{F} = 2$, then we denote by \mathbb{K} the Artin–Schreier subspace $\mathbb{K} = \{\delta + \delta^2 \mid \delta \in \mathbb{F}\}$. In the descriptions of the algebras in characteristic 3 in $(4/3)$, we use a subspace \mathbb{K}_β defined as $\mathbb{K}_\beta = \{\beta\delta^3 + \delta \mid \delta \in \mathbb{F}\}$.

We note that if $\langle x_1, \dots, x_k \rangle$ is a basis for a Lie algebra L , then any $[p]$ -map on L is determined by the images $x_1^{[p]}, \dots, x_k^{[p]}$.

Theorem 2.1. *Suppose that L is a nilpotent Lie algebra of dimension at most 4 over a perfect field \mathbb{F} . Then the equivalence classes of the $[p]$ -maps on L are as follows.*

(1/1) If $\dim L = 1$ and $L = \langle x_1 \rangle$:

(a) $x_1^{[p]} = 0$.

(1/1) If $L = \langle x_1, x_2 \rangle$:

(a) $x_1^{[p]} = x_2^{[p]} = 0$;

(b) $x_1^{[p]} = x_2, x_2^{[p]} = 0$.

(3/1) If $L = \langle x_1, x_2, x_3 \rangle$:

(a) $x_1^{[p]} = x_2^{[p]} = x_3^{[p]} = 0$;

(b) $x_1^{[p]} = x_2, x_2^{[p]} = x_3^{[p]} = 0$;

(c) $x_1^{[p]} = x_2, x_2^{[p]} = x_3, x_3^{[p]} = 0$.

(3/2) Suppose that $L = \langle x_1, x_2, x_3 \mid [x_1, x_2] = x_3 \rangle$. If $\text{char } \mathbb{F} \geq 3$:

(a) $x_1^{[p]} = x_2^{[p]} x_3^{[p]} = 0$;

(b) $x_1^{[p]} = x_3, x_2^{[p]} = x_3^{[p]} = 0$.

If $\text{char } \mathbb{F} = 2$:

(a) $x_1^{[2]} = x_3, x_2^{[2]} = \xi x_3, x_3^{[2]} = 0$.

The parameters ξ_1 and ξ_2 result in equivalent $[p]$ -maps if and only if $\xi_1 + \xi_2 \in \mathbb{K}$.

(4/1) If $L = \langle x_1, x_2, x_3, x_4 \rangle$:

(a) $x_1^{[p]} = x_2^{[p]} = x_3^{[p]} = x_4^{[p]} = 0$;

(b) $x_1^{[p]} = x_2, x_1^{[p]} = x_2^{[p]} = x_3^{[p]} = 0$;

(c) $x_1^{[p]} = x_2, x_3^{[p]} = x_4, x_2^{[p]} = x_4^{[p]} = 0$;

(d) $x_1^{[p]} = x_2, x_2^{[p]} = x_3, x_3^{[p]} = x_4^{[p]} = 0$;

(e) $x_1^{[p]} = x_2, x_2^{[p]} = x_3, x_3^{[p]} = x_4, x_4^{[p]} = 0$.

(4/2) Suppose that $L = \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2] = x_3 \rangle$. If $\text{char } \mathbb{F} \geq 3$:

(a) $x_1^{[p]} = x_2^{[p]} = x_3^{[p]} = x_4^{[p]} = 0$;

(b) $x_1^{[p]} = x_3, x_2^{[p]} = x_3^{[p]} = x_4^{[p]} = 0$;

(c) $x_1^{[p]} = x_4, x_2^{[p]} = x_3^{[p]} = x_4^{[p]} = 0$;

(d) $x_1^{[p]} = x_3, x_2^{[p]} = x_4, x_3^{[p]} = x_4^{[p]} = 0$;

(e) $x_3^{[p]} = x_4, x_1^{[p]} = x_2^{[p]} = x_4^{[p]} = 0$;

(f) $x_3^{[p]} = x_4, x_2^{[p]} = x_3, x_1^{[p]} = x_4^{[p]} = 0$;

(g) $x_4^{[p]} = x_3, x_1^{[p]} = x_2^{[p]} = x_3^{[p]} = 0$;

(h) $x_4^{[p]} = x_3, x_2^{[p]} = x_4, x_1^{[p]} = x_3^{[p]} = 0$.

If $\text{char } \mathbb{F} = 2$:

(a) $x_1^{[2]} = x_3, x_2^{[2]} = \xi x_3, x_3^{[2]} = x_4^{[2]} = 0$;

(b) $x_1^{[2]} = x_4, x_2^{[2]} = x_3^{[2]} = x_4^{[2]} = 0$;

(c) $x_1^{[2]} = x_3, x_2^{[2]} = x_4$;

(d) $x_3^{[2]} = x_4, x_1^{[2]} = x_3, x_2^{[2]} = \xi x_3$;

(e) $x_4^{[2]} = x_3, x_1^{[2]} = x_2^{[2]} = x_3^{[2]} = 0$;

(f) $x_4^{[2]} = x_3, x_2^{[2]} = x_4, x_1^{[2]} = x_3^{[2]} = 0$.

In cases (a) and (d), the parameters ξ_1, ξ_2 represent equivalent $[p]$ -maps if and only if $\xi_1 + \xi_2 \in \mathbb{K}$.

(4/3) Suppose that $L = \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4 \rangle$. If $\text{char } \mathbb{F} \geq 5$:

- (a) $x_1^{[p]} = x_2^{[p]} = x_3^{[p]} = x_4^{[p]} = 0$;
- (b) $x_1^{[p]} = x_4, x_2^{[p]} = x_3^{[p]} = x_4^{[p]} = 0$;
- (c) $x_2^{[p]} = \xi x_4, x_1^{[p]} = x_3^{[p]} = x_4^{[p]} = 0$;
- (d) $x_3^{[p]} = x_4, x_1^{[p]} = x_2^{[p]} = x_4^{[p]} = 0$.

The parameters ξ_1 and ξ_2 represent isomorphic algebras if and only if $\xi_1 \xi_2^{-1}$ is a square in \mathbb{F} .

If $\text{char } \mathbb{F} = 3$:

- (a) $x_1^{[3]} = x_2^{[3]} = x_3^{[3]} = x_4^{[3]} = 0$;
- (b) $x_3^{[3]} = x_4, x_1^{[2]} = x_2^{[3]} = x_4^{[3]} = 0$;
- (c) $x_1^{[3]} = \alpha x_4, x_2^{[3]} = \beta x_4, x_3^{[3]} = x_4^{[3]} = 0$.

Where $\alpha \in \mathbb{F}, \beta \in \mathbb{F}^*$ and the pairs $(\alpha_1, \beta_1), (\alpha_2, \beta_2)$ represent equivalent $[p]$ -maps if and only if β_2/β_1 is a square in F and $\alpha_1 \pm \alpha_2 \sqrt{\beta_2/\beta_1} \in \mathbb{K}_{\beta_1}$. If $\text{char } \mathbb{F} = 2$, then L is not restrictable.

Through out the paper, we assume that \mathbb{F} is a perfect field of characteristic p and unless otherwise stated $p \geq 3$. Thus, the Frobenius automorphism of \mathbb{F} given by $x \mapsto x^p$ is invertible and we denote the inverse image of x by $x^{1/p}$.

3. ABELIAN LIE ALGEBRAS

In this section we classify abelian $[p]$ -nilpotent restricted Lie algebras. In dimension 1, the only $[p]$ -nilpotent restricted Lie algebra is $\langle x \mid x^{[p]} = 0 \rangle$. Suppose that $\dim L = 2$. It is not hard to see that there are two possible $[p]$ -maps on L that result in abelian $[p]$ -nilpotent Lie algebras. More precisely, there exist linearly independent elements $x_1, x_2 \in L$ such that either $x_1^{[p]} = x_2^{[p]} = 0$ or $x_1^{[p]} = x_2$ and $x_2^{[p]} = 0$.

Suppose now that L is abelian with $\dim L = 3$ and let $\{x_1, x_2, x_3\}$ be a basis of L . Since L is $[p]$ -nilpotent, without loss of generality we assume that $x_3^{[p]} = 0$. Let $H = L/\langle x_3 \rangle$. Since $\dim H = 2$, we may assume by the argument of the first paragraph of the section that either $x_1^{[p]}, x_2^{[p]} \in \langle x_3 \rangle$ or $x_1^{[p]} = x_2, x_2^{[p]} \in \langle x_3 \rangle$.

First consider the case $x_1^{[p]}, x_2^{[p]} \in \langle x_3 \rangle$. Hence $x_1^{[p]} = \alpha x_3$ and $x_2^{[p]} = \beta x_3$, for some $\alpha, \beta \in \mathbb{F}$. If $\alpha \neq 0$ then we rescale x_1 so that $x_1^{[p]} = x_3$. Similarly, if $\beta \neq 0$ then we rescale x_2 so that $x_2^{[p]} = x_3$. If $x_1^{[p]} = x_3$ and $x_2^{[p]} = x_3$ then $(-x_1 + x_2)^{[p]} = 0$. In this case, we replace x_2 with $-x_1 + x_2$ to obtain $x_2^{[p]} = 0$. Thus we conclude that, up to isomorphism, the possible $[p]$ -maps on L are as follows:

$$\begin{aligned} x_1^{[p]} &= x_2^{[p]} = x_3^{[p]} = 0; \\ x_1^{[p]} &= x_3, x_2^{[p]} = x_3^{[p]} = 0. \end{aligned}$$

Now consider the case that $x_1^{[p]} - x_2, x_2^{[p]} \in \langle x_3 \rangle$. Hence $x_1^{[p]} = x_2 + \alpha x_3$ and $x_2^{[p]} = \beta x_3$, for some $\alpha, \beta \in \mathbb{F}$. We replace x_2 with $x_2 + \alpha x_3$ to obtain $x_1^{[p]} = x_2$. If $\beta \neq 0$ then we rescale x_3 so that $x_2^{[p]} = x_3$. We conclude that, up to isomorphism, possible $[p]$ -maps on L are as follows:

$$\begin{aligned} x_1^{[p]} &= x_2, \quad x_2^{[p]} = x_3^{[p]} = 0; \\ x_1^{[p]} &= x_2, \quad x_2^{[p]} = x_3, \quad x_3^{[p]} = 0. \end{aligned}$$

The first algebra is isomorphic to one of the algebras above. Thus, up to isomorphism, 3-dimensional abelian $[p]$ -nilpotent restricted Lie algebras are as follows:

$$\begin{aligned} L_{3,1}^1 &= \langle x_1, x_2, x_3 \mid x_1^{[p]} = x_2^{[p]} = x_3^{[p]} = 0 \rangle; \\ L_{3,1}^2 &= \langle x_1, x_2, x_3 \mid x_1^{[p]} = x_2, \quad x_2^{[p]} = x_3^{[p]} = 0 \rangle; \\ L_{3,1}^3 &= \langle x_1, x_2, x_3 \mid x_1^{[p]} = x_2, \quad x_2^{[p]} = x_3, \quad x_3^{[p]} = 0 \rangle. \end{aligned}$$

Let x_1, x_2, x_3, x_4 be a basis of a 4-dimensional abelian Lie algebra $L = L_{4,1}$. Since L is $[p]$ -nilpotent, without loss of generality we assume that $x_4^{[p]} = 0$. We set $H = L/\langle x_4 \rangle$ and perform similar calculations as in the lower-dimensional cases to show that, up to isomorphism, 4-dimensional abelian $[p]$ -nilpotent restricted Lie algebras are as follows:

$$\begin{aligned} L_{4,1}^1 &= \langle x_1, x_2, x_3, x_4 \mid x_1^{[p]} = x_2^{[p]} = x_3^{[p]} = x_4^{[p]} = 0 \rangle; \\ L_{4,1}^2 &= \langle x_1, x_2, x_3, x_4 \mid x_1^{[p]} = x_2, \quad x_2^{[p]} = x_3^{[p]} = x_4^{[p]} = 0 \rangle; \\ L_{4,1}^3 &= \langle x_1, x_2, x_3, x_4 \mid x_1^{[p]} = x_2, \quad x_3^{[p]} = x_4, \quad x_2^{[p]} = x_4^{[p]} = 0 \rangle; \\ L_{4,1}^4 &= \langle x_1, x_2, x_3, x_4 \mid x_1^{[p]} = x_2, \quad x_2^{[p]} = x_3, \quad x_3^{[p]} = x_4^{[p]} = 0 \rangle; \\ L_{4,1}^5 &= \langle x_1, x_2, x_3, x_4 \mid x_1^{[p]} = x_2, \quad x_2^{[p]} = x_3, \quad x_3^{[p]} = x_4, \quad x_4^{[p]} = 0 \rangle. \end{aligned}$$

4. THE HEISENBERG LIE ALGEBRA

Consider the Heisenberg Lie algebra

$$L = \langle x_1, x_2, x_3 \mid [x_1, x_2] = x_3 \rangle.$$

Since $(\text{ad } x)^p = 0$ for all $p \geq 2$, the image of a $[p]$ -map on L is in $Z(L)$. Let $\varphi : L \rightarrow Z(L)$ be a $[p]$ -map from L to $Z(L) = \langle x_3 \rangle$. Then φ can be described by the images of x_1, x_2 , and x_3 . The fact that (L, φ) is $[p]$ -nilpotent implies that $x_3\varphi = 0$. Hence φ is described by a vector (α, β) where

$$x_1\varphi = \alpha x_3 \quad \text{and} \quad x_2\varphi = \beta x_3.$$

The automorphism group of L , acting on row vectors with respect to the given basis, consists of the invertible 3×3 -matrices of the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & d \end{pmatrix}$$

where $d = a_{11}a_{22} - a_{12}a_{21}$. Let A be an automorphism as above and let us compute the vector (α', β') that describes $A\varphi A^{-1}$.

4.1. Odd characteristic. Let us first assume that $p \geq 3$. In this case the $[p]$ -map φ is semilinear. Therefore

$$x_1 A \varphi A^{-1} = (a_{11}x_1 + a_{12}x_2 + a_{13}x_3) \varphi A^{-1} = (a_{11}^p \alpha + a_{12}^p \beta) x_3 A^{-1} = (a_{11}^p \alpha + a_{12}^p \beta) d^{-1} x_3.$$

Hence $\alpha' = (a_{11}^p \alpha + a_{12}^p \beta) d^{-1}$ and we obtain similarly that $\beta' = (a_{21}^p \alpha + a_{22}^p \beta) d^{-1}$. We claim in this case that L must be isomorphic to one of the following algebras.

$$\begin{aligned} L_{3,2}^1 &= \langle x_1, x_2, x_3 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_2^{[p]} = x_3^{[p]} = 0 \rangle; \\ L_{3,2}^2 &= \langle x_1, x_2, x_3 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_3, x_2^{[p]} = x_3^{[p]} = 0 \rangle. \end{aligned}$$

The algebras $L_{3,2}^1$ and $L_{3,2}^2$ are clearly non-isomorphic, as $(L_{3,2}^1)^{[p]} = 0$ while $(L_{3,2}^2)^{[p]} = \langle x_3 \rangle$.

Let $\varphi : L \rightarrow Z(L)$ be a semilinear transformation. Then $x_3 \varphi = 0$, $x_1 \varphi = \alpha x_3$, and $x_2 \varphi = \beta x_3$, for some $\alpha, \beta \in \mathbb{F}$. If $\alpha = \beta = 0$ then we have $L_{3,2}^1$. Suppose that this is not the case and assume without loss of generality that $\alpha \neq 0$. Note that $(cx_1 + x_2) \varphi = (c^p \alpha + \beta) x_3$, for every $c \in \mathbb{F}$. Hence choosing $c_0 = -(\beta/\alpha)^{1/p}$ and $x'_2 = c_0 x_1 + x_2$, we obtain $x'_2 \varphi = 0$. Then replacing x_3 with αx_3 and x_2 with αx_2 we find that $x_1 \varphi = x_3$ and hence we obtain $L_{3,2}^2$.

4.2. Characteristic 2. Suppose that L is the Heisenberg Lie algebra over a field \mathbb{F} of characteristic 2. In this section we classify the possible $[p]$ -nilpotent $[p]$ -maps on L . First we set

$$(1) \quad \mathbb{K} = \{ \delta + \delta^2 \mid \delta \in \mathbb{F} \}.$$

As the characteristic is 2, \mathbb{K} is an \mathbb{F}_2 -subspace of \mathbb{F} . In fact, \mathbb{K} is the image of the \mathbb{F}_2 -linear map $\delta \mapsto \delta + \delta^2$ whose kernel is \mathbb{F}_2 . Hence if \mathbb{F} is a finite field, \mathbb{K} has co-dimension 1, but this may not be true for other fields. For instance if \mathbb{F} is algebraically closed, then the polynomial $t^2 + t + \alpha$ has a root for any $\alpha \in \mathbb{F}$ and so $\mathbb{K} = \mathbb{F}$. The subspace \mathbb{K} is often referred to as the Artin-Schreier subspace; see for instance [4, page 70].

We claim that L is isomorphic to a Lie algebra of the form

$$K_{3,2}^1(\xi) = \langle x_1, x_2, x_3 \mid [x_1, x_2] = x_3, x_1^{[2]} = x_3, x_2^{[2]} = \xi x_3, x_3^{[2]} = 0 \rangle,$$

where $\xi \in \mathbb{F}$. Further, $K_{3,2}^1(\xi_1) \cong K_{3,2}^1(\xi_2)$ if and only if $\xi_1 + \xi_2 \in \mathbb{K}$.

Note that the map $\varphi : L \rightarrow L$ defined by $x \mapsto x^{[2]}$ is not semilinear. However, since $0 = (\text{ad } x_1)^2 = \text{ad } x_1^{[2]}$ and $0 = (\text{ad } x_2)^2 = \text{ad } x_2^{[2]}$, we have that $\varphi : L \rightarrow Z(L)$. Suppose that φ is the $[p]$ -map represented by the vector (α, β) as in the odd characteristic case. Then computing $A\varphi A^{-1}$ for an automorphism $A = (a_{i,j})$ we obtain that $A\varphi A^{-1}$ is represented by the vector (α', β') where

$$\begin{aligned}\alpha' &= d^{-1}(\alpha a_{11}^2 + \beta a_{12}^2 + a_{11}a_{12}) \\ \beta' &= d^{-1}(\alpha a_{21}^2 + \beta a_{22}^2 + a_{21}a_{22})\end{aligned}$$

where $d = a_{11}a_{22} + a_{12}a_{21}$. First we show that every Lie algebra is isomorphic to $L_{3,2}^1(\xi)$ with some $\xi \in \mathbb{F}$. If $\alpha = \beta = 0$, then replace x_1 with $x_1 + x_2$ to obtain $x_1^{[2]} = x_3$. If $\alpha \neq 0$ then replace x_2 and x_3 with αx_2 and αx_3 , respectively, to obtain $x_1^{[2]} = x_3$. If $\alpha = 0$, but $\beta \neq 0$, then swap x_1 and x_2 and repeat the steps in the previous sentence. Hence L is isomorphic to $L_{3,2}^1(\xi)$ with some $\xi \in \mathbb{F}$, as claimed.

Let us now show the claim concerning the isomorphisms between the algebras $K_{3,2}^1(\xi)$. Suppose, for $i = 1, 2$, that φ_i is represented by the vector $(1, \xi_i)$. First if $\xi_1 + \xi_2 \in \mathbb{K}$, then choose $\delta \in \mathbb{F}$ such that $\xi_1 + \delta^2 + \delta = \xi_2$ and consider the automorphism A with $a_{11} = 1$, $a_{12} = 0$, $a_{21} = \delta$ and $a_{22} = 1$. Then $A\varphi_1 A^{-1} = \varphi_2$. Therefore $L_{3,2}^1(\xi_1) \cong L_{3,2}^1(\xi_2)$.

Conversely, suppose that $K_{3,2}^1(\xi_1) \cong K_{3,2}^1(\xi_2)$, that is, there is some $A \in \text{Aut}(L)$ such that $A\varphi_1 A^{-1} = \varphi_2$. First we note that diagonal automorphisms of the form $\text{diag}(a, a, a^2)$ stabilize φ_1 for all $a \in \mathbb{F}^*$. Suppose that $a_{22} = 0$. Let $d = a_{11}a_{22} + a_{12}a_{21} = a_{12}a_{21}$. Then swapping A with $\text{diag}(d^{-1/2}, d^{-1/2}, d^{-1})A$, we may, and will, assume that $a_{12}a_{21} = d = 1$ and hence $a_{21} = a_{12}^{-1}$. Thus $A\varphi_1 A^{-1} = \varphi_2$ implies that

$$\begin{aligned}a_{11}^2 + \xi_1 a_{12}^2 + a_{11}a_{12} &= 1; \\ a_{12}^{-2} &= \xi_2.\end{aligned}$$

Combining these equations, we find $\xi_1 + \xi_2 = (a_{11}/a_{12})^2 + a_{11}/a_{12} \in \mathbb{K}$, as required.

Assume now $a_{22} \neq 0$ and $d = a_{11}a_{22} + a_{12}a_{21}$. We may suppose, as above, that $d = a_{11}a_{22} + a_{12}a_{21} = 1$. Thus we obtain

$$\begin{aligned}(2) \quad & a_{11}^2 + \xi_1 a_{12}^2 + a_{11}a_{12} = 1; \\ (3) \quad & a_{11}a_{22} + a_{12}a_{21} = 1; \\ (4) \quad & a_{21}^2 + \xi_1 a_{22}^2 + a_{21}a_{22} = \xi_2.\end{aligned}$$

Equation (3) implies that $a_{11} = (1 + a_{12}a_{21})/a_{22}$ which we substitute into equation (2) to obtain that

$$1 + a_{12}^2 a_{21}^2 + \xi_1 a_{12}^2 a_{22}^2 + a_{22}a_{12} + a_{22}a_{12}^2 a_{21} = a_{22}^2$$

and hence

$$(5) \quad a_{22}^2 = 1 + a_{12}^2(a_{21}^2 + \xi_1 a_{22}^2 + a_{22}a_{21}) + a_{22}a_{12} = 1 + a_{12}^2 \xi_2 + a_{22}a_{12}.$$

If $a_{12} = 0$ then this implies that $a_{22} = 1$, and then equation (4) gives that $\xi_1 + \xi_2 \in \mathbb{K}$ as required. If $a_{12} \neq 0$, then equation (5) gives that $\xi_2 + a_{12}^{-2} \in \mathbb{K}$. On the other hand, in this case, equation (2) gives that $\xi_1 + a_{12}^{-2} \in \mathbb{K}$ and hence $\xi_1 + \xi_2 \in \mathbb{K}$, as claimed.

5. RESTRICTION MAPS ON $L_{4,2}$

Consider the Lie algebra

$$L = L_{4,2} = \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2] = x_3 \rangle.$$

The automorphism group of L consists of the set of invertible matrices of the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & d & 0 \\ 0 & 0 & a_{43} & a_{44} \end{pmatrix},$$

where $d = a_{11}a_{22} - a_{12}a_{21}$. Note that $Z(L) = \langle x_3, x_4 \rangle$ has dimension 2. Since L is $[p]$ -nilpotent, we have

$$Z(L)^{[p]^2} < Z(L)^{[p]} < Z(L).$$

In particular, $Z(L)^{[p]^2} = 0$. Since the nilpotency class of L is two, a restricted Lie algebra structure on L is given by a semilinear transformation from L to $Z(L)$ whenever $p \geq 3$.

First we sort out the possible restrictions of a $[p]$ -map on $Z(L)$, for every $p \geq 2$. Suppose that $x_3^{[p]} \neq 0$. Then $x_3^{[p]} = \alpha x_3 + \beta x_4$. If $\beta = 0$, then $x_3^{[p]} = \alpha x_3$, and so $x_3^{[p]^n} \neq 0$ which contradicts the assumption that L is $[p]$ -nilpotent. Hence $\beta \neq 0$ and we may replace x_4 with $x'_4 = \alpha x_3 + \beta x_4$. This replacement is an automorphism of L and then we get $x_3^{[p]} = x_4$. Now $x_4^{[p]} = \gamma x_3 + \delta x_4$, which gives that

$$0 = x_4^{[p]^2} = (\gamma x_3 + \delta x_4)^{[p]} = (\gamma^p + \delta^p \delta) x_4 + \delta^p \gamma x_3.$$

Hence, $\delta = \gamma = 0$ and so $x_4^{[p]} = 0$.

If $x_3^{[p]} = 0$ and $x_4^{[p]} \neq 0$, then $x_4^{[p]} = \alpha x_3 + \beta x_4$. We have:

$$0 = x_4^{[p]^2} = (\alpha x_3 + \beta x_4)^{[p]} = \beta^p x_4^{[p]}$$

which shows that $\beta = 0$ and hence $x_4^{[p]} = \alpha x_3$. Replacing x_4 with $(1/\alpha)^{1/p} x_4$ is a Lie algebra automorphism and results in $x_4^{[p]} = x_3$.

Thus, the map φ on $Z(L)$ can have three different forms: either $x_3^{[p]} = x_4^{[p]} = 0$; or $x_3^{[p]} = x_4$ and $x_4^{[p]} = 0$; or $x_3^{[p]} = 0$ and $x_4^{[p]} = x_3$. We will consider these three cases separately.

5.1. Suppose first that $x_3^{[p]} = x_4^{[p]} = 0$ and let φ be a semilinear map that extends this $[p]$ -map to the whole L . Let $x_1\varphi = \alpha_1 x_3 + \beta_1 x_4$ and $x_2\varphi = \alpha_2 x_3 + \beta_2 x_4$. Let $A \in \text{Aut}(L)$ and consider the semilinear transformation $\varphi' = A\varphi A^{-1}$ given by $x_1\varphi' = \alpha'_1 x_3 + \beta'_1 x_4$ and $x_2\varphi' = \alpha'_2 x_3 + \beta'_2 x_4$. Let us compute the coefficients $\alpha'_1, \alpha'_2, \beta'_1, \beta'_2$. First, we note that

$$x_3 A^{-1} = d^{-1} x_3 \quad \text{and} \quad x_4 A^{-1} = -a_{43}/(da_{44}) x_3 + a_{44}^{-1} x_4.$$

Now

$$\begin{aligned}
x_1 A \varphi A^{-1} &= (a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4) \varphi A^{-1} \\
&= ((\alpha_1 a_{11}^p + \alpha_2 a_{12}^p)x_3 + (\beta_1 a_{11}^p + \beta_2 a_{12}^p)x_4) A^{-1} \\
&= \left(d^{-1}(\alpha_1 a_{11}^p + \alpha_2 a_{12}^p) - \frac{a_{43}}{da_{44}}(\beta_1 a_{11}^p + \beta_2 a_{12}^p) \right) x_3 + a_{44}^{-1}(\beta_1 a_{11}^p + \beta_2 a_{12}^p) x_4.
\end{aligned}$$

Repeating the calculation for $x_2 A \varphi A^{-1}$, we obtain that

$$\begin{aligned}
\alpha'_1 &= d^{-1}(\alpha_1 a_{11}^p + \alpha_2 a_{12}^p) - \frac{a_{43}}{da_{44}}(\beta_1 a_{11}^p + \beta_2 a_{12}^p) \\
\alpha'_2 &= d^{-1}(\alpha_1 a_{21}^p + \alpha_2 a_{22}^p) - \frac{a_{43}}{da_{44}}(\beta_1 a_{21}^p + \beta_2 a_{22}^p) \\
\beta'_1 &= a_{44}^{-1}(\beta_1 a_{11}^p + \beta_2 a_{12}^p) \\
\beta'_2 &= a_{44}^{-1}(\beta_1 a_{21}^p + \beta_2 a_{22}^p).
\end{aligned}$$

Thus

$$(6) \quad (\alpha'_1, \alpha'_2, \beta'_1, \beta'_2) = (\alpha_1, \alpha_2, \beta_1, \beta_2) \begin{pmatrix} d^{-1}a_{11}^p & d^{-1}a_{21}^p & 0 & 0 \\ d^{-1}a_{12}^p & d^{-1}a_{22}^p & 0 & 0 \\ -\frac{a_{43}}{da_{44}}a_{11}^p & -\frac{a_{43}}{da_{44}}a_{21}^p & a_{44}^{-1}a_{11}^p & a_{44}^{-1}a_{21}^p \\ -\frac{a_{43}}{da_{44}}a_{12}^p & -\frac{a_{43}}{da_{44}}a_{22}^p & a_{44}^{-1}a_{12}^p & a_{44}^{-1}a_{22}^p \end{pmatrix}.$$

Hence, the action of the automorphism in (6) on the vector space \mathbb{F}^4 can be described by the tensor product

$$(7) \quad \begin{pmatrix} d^{-1} & 0 \\ -\frac{a_{43}}{da_{44}} & a_{44}^{-1} \end{pmatrix} \otimes \begin{pmatrix} a_{11}^p & a_{21}^p \\ a_{12}^p & a_{22}^p \end{pmatrix}$$

acting on the tensor product $V_1 \otimes V_2$, where $V_1 = V_2 = \mathbb{F}^2$. Let us calculate the orbits under this action. We denote the group of matrices of the form (7) by H . Let e_1, e_2 and f_1, f_2 be the standard bases of V_1 and V_2 , respectively.

Let $W = V_1 \otimes V_2$ and let $v \in W \setminus 0$. First suppose that $v = v_1 \otimes v_2$ with $v_1 \in V_1 \setminus 0$ and $v_2 \in V_2 \setminus 0$. Since $\text{GL}(2, \mathbb{F})$ is transitive on the non-zero vectors of V_2 , there exists $g_2 \in \text{GL}(2, \mathbb{F})$ such that $v_2 g_2 = (1, 0)$. Choose g_1 such that $g_1 \otimes g_2 \in H$. Then

$$(v_1 \otimes v_2)(g_1 \otimes g_2) = v'_1 \otimes (1, 0).$$

Let us now consider vectors of the form $(\alpha, \beta) \otimes (1, 0)$. If $\beta = 0$ we have

$$((\alpha, 0) \otimes (1, 0)) \left(\begin{pmatrix} \alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & \alpha^p \end{pmatrix} \right) = (1, 0) \otimes (1, 0).$$

If $\beta \neq 0$ then

$$((\alpha, \beta) \otimes (1, 0)) \left(\begin{pmatrix} 1 & 0 \\ -\alpha/\beta & \beta^{-1} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) = (0, 1) \otimes (1, 0).$$

We deduce that the group H has three orbits on the set of pure tensors with orbit representatives $0, (1, 0) \otimes (1, 0)$ and $(0, 1) \otimes (1, 0)$.

Let us compute the orbits of H on the set of elements that are not pure tensors. Such an element is of the form $e_1 \otimes v_1 + e_2 \otimes v_2$ with v_1 and v_2 linearly independent. Note that there exists a $g^p \in \mathbf{GL}(2, \mathbb{F})$ that maps $v_1 \mapsto f_1$ and $v_2 \mapsto f_2$. Let $d = \det g$. Then

$$(e_1 \otimes v_1 + e_2 \otimes v_2) \left(\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \otimes g \right) = de_1 \otimes f_1 + e_2 \otimes f_2.$$

Hence every orbit contains an element of the form $\alpha e_1 \otimes f_1 + e_2 \otimes f_2$ with $\alpha \in \mathbb{F}^*$. Now

$$(\alpha e_1 \otimes f_1 + e_2 \otimes f_2) \left(\begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha^{-p} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & \alpha^p \end{pmatrix} \right) = e_1 \otimes f_1 + e_2 \otimes f_2.$$

Hence, H has 4 orbits and the representatives of these orbits are 0, $(1, 0) \otimes (1, 0)$, $(0, 1) \otimes (1, 0)$, and $(1, 0) \otimes (1, 0) + (0, 1) \otimes (0, 1)$. The corresponding restricted Lie algebras are

$$\begin{aligned} L_{4,2}^1 &= \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2] = x_3 \rangle; \\ L_{4,2}^2 &= \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_3 \rangle; \\ L_{4,2}^3 &= \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_4 \rangle; \\ L_{4,2}^4 &= \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2] = x_3, x_1^{[p]} = x_3, x_2^{[p]} = x_4 \rangle. \end{aligned}$$

5.1.1. *Characteristic 2.* Let us now assume that $x_3^{[2]} = x_4^{[2]} = 0$ and $\text{char } \mathbb{F} = 2$. In this case we will show that L is isomorphic to one of the following Lie algebras:

$$\begin{aligned} K_{4,2}^1(\xi) &= \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2] = x_3, x_1^{[2]} = x_3, x_2^{[2]} = \xi x_3 \rangle; \\ K_{4,2}^2 &= \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2] = x_3, x_1^{[2]} = x_4 \rangle; \\ K_{4,2}^3 &= \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2] = x_3, x_1^{[2]} = x_3, x_2^{[2]} = x_4 \rangle. \end{aligned}$$

We claim further that $K_{4,2}^1(\xi_1) \cong K_{4,2}^1(\xi_2)$ if and only if $\xi_1 + \xi_2 \in \mathbb{K}$ where \mathbb{K} is the Artin-Schreier subspace defined in (1).

First assume that $L^{[2]} \leq \langle x_3 \rangle$. In this case $L = L_1 \oplus \langle x_4 \rangle$ where $L_1 = \langle x_1, x_2, x_3 \rangle$ and the direct sum is interpreted as a direct sum of restricted ideals. Hence L is isomorphic to an algebra of the form $K_{3,2}^1(\xi) \oplus \mathbb{F}$ where $K_{3,2}^1(\xi)$ is an algebra defined in Section 4.2. Thus $L \cong K_{4,2}^1(\xi)$ with some $\xi \in \mathbb{F}$.

Hence we may assume that $L^{[2]} \not\leq \langle x_3 \rangle$. Assume that $x_1^{[2]} = \alpha_1 x_3 + \beta_1 x_4$ and that $x_2^{[2]} = \alpha_2 x_3 + \beta_2 x_4$. Either by swapping x_1 and x_2 or replacing x_1 with $x_1 + x_2$, we may assume without loss of generality that $x_1^{[2]} \neq 0$. Assume first that $x_2^{[2]} = 0$. As $L^{[2]} \not\leq \langle x_3 \rangle$, we must have $\beta_1 \neq 0$. Then replace x_4 with $\alpha_1 x_3 + \beta_1 x_4$ to obtain that $L \cong K_{4,2}^2$.

Suppose now that $x_1^{[2]} \neq 0$ and $x_2^{[2]} \neq 0$. If $\beta_2 \neq 0$, then we replace x_4 with $\alpha_2 x_3 + \beta_2 x_4$ and obtain that $x_2^{[2]} = x_4$. Now $(x_1 + \beta_1^{1/2} x_2)^{[2]} = \alpha_1 x_3$. Hence replacing x_1 with $x_1 + \beta_1^{1/2} x_2$

we may assume that $x_1^{[2]} = \alpha_1 x_3$ with some $\alpha_1 \in \mathbb{F}^*$. Now replace x_2 by $\alpha_1 x_2$, x_3 by $\alpha_1 x_3$ and x_4 by $\alpha_1^2 x_4$ to obtain that $x_1^{[2]} = x_3$ and $x_2^{[2]} = x_4$ and hence $L \cong K_{4,2}^3$. If $\beta_1 \neq 0$ then swapping x_1 and x_2 allows us to repeat this argument.

These Lie algebras are pairwise non-isomorphic, as $K_{4,2}^1(\xi)$ are the only ones with $L^{[2]} = L'$, $K_{4,2}^2$ is the only one with $L^{[2]} \cap L' = 0$, while $K_{4,2}^3$ is the only one with $L' < L^{[2]}$. The claim concerning the isomorphisms among the algebras $K_{4,2}^1(\xi)$ follows from the fact that if L is such an algebra and I is a one-dimensional ideal, then $I \leq Z(L) = \langle x_3, x_4 \rangle$, and so L/I is either abelian (if and only if $I = \langle x_3 \rangle$) or is isomorphic to $K_{3,2}^1(\xi)$ in Section 4.2.

5.2. Let us now consider the case when $x_3^{[p]} = x_4$ and $x_4^{[p]} = 0$. First we note that if A is an automorphism of L then A preserves $\varphi|_{Z(L)}$ if and only if $a_{43} = 0$ and $d^p = a_{44}$. Let $x_1\varphi = \alpha_1 x_3 + \beta_1 x_4$ and $x_2\varphi = \alpha_2 x_3 + \beta_2 x_4$. Now consider the semilinear transformation $\varphi' = A\varphi A^{-1}$ given by $x_1\varphi' = \alpha'_1 x_3 + \beta'_1 x_4$ and $x_2\varphi' = \alpha'_2 x_3 + \beta'_2 x_4$.

5.2.1. *Odd characteristic.* Similar calculations as in Section 5.1 show that

$$\begin{aligned}\alpha'_1 &= d^{-1}(\alpha_1 a_{11}^p + \alpha_2 a_{12}^p) \\ \alpha'_2 &= d^{-1}(\alpha_1 a_{21}^p + \alpha_2 a_{22}^p) \\ \beta'_1 &= a_{44}^{-1}(\beta_1 a_{11}^p + \beta_2 a_{12}^p + a_{13}^p) \\ \beta'_2 &= a_{44}^{-1}(\beta_1 a_{21}^p + \beta_2 a_{22}^p + a_{23}^p)\end{aligned}$$

Let us write

$$(\alpha_1, \alpha_2, \beta_1, \beta_2)(A\varrho) = (\alpha'_1, \alpha'_2, \beta'_1, \beta'_2)$$

Thus, in matrix form we have:

$$(\alpha_1, \alpha_2, \beta_1, \beta_2)(A\varrho) = (\alpha_1, \alpha_2, \beta_1, \beta_2) \begin{pmatrix} d^{-1}a_{11}^p & d^{-1}a_{21}^p & 0 & 0 \\ d^{-1}a_{12}^p & d^{-1}a_{22}^p & 0 & 0 \\ 0 & 0 & a_{44}^{-1}a_{11}^p & a_{44}^{-1}a_{21}^p \\ 0 & 0 & a_{44}^{-1}a_{12}^p & a_{44}^{-1}a_{22}^p \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ a_{44}^{-1}a_{13}^p \\ a_{44}^{-1}a_{23}^p \end{pmatrix}.$$

We claim that there are two orbits with representatives $(0, 0, 0, 0)$ and $(0, 1, 0, 0)$. First notice that

$$(\alpha_1, \alpha_2, \beta_1, \beta_2) \begin{pmatrix} 1 & 0 & a_{13} & 0 \\ 0 & 1 & a_{23} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \varrho = (\alpha_1, \alpha_2, \beta_1 + a_{13}^p, \beta_2 + a_{23}^p).$$

Since a_{13} and a_{23} can be chosen freely, $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ and $(\alpha_1, \alpha_2, \beta_3, \beta_4)$ are in the same orbit for all $\beta_1, \beta_2, \beta_3, \beta_4 \in \mathbb{F}$. Let $(\alpha_1, \alpha_2, \beta_1, \beta_2) \in \mathbb{F}^4$. If $\alpha_1 = \alpha_2 = 0$, then the set of such elements would form a single orbit with orbit representative $(0, 0, 0, 0)$. Suppose now that $(\alpha_1, \alpha_2) \neq (0, 0)$. We may assume without loss of generality that $\beta_1 = \beta_2 = 0$. If

$\alpha_2 \neq 0$ then

$$(\alpha_1, \alpha_2, 0, 0) \begin{pmatrix} \alpha_2^{1/p} & -\alpha_1^{1/p} & 0 & 0 \\ 0 & \alpha_2^{-1/p} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \varrho = (0, 1, 0, 0).$$

If $\alpha_2 = 0$ then $\alpha_1 \neq 0$ and we obtain that

$$(\alpha_1, 0, 0, 0) \begin{pmatrix} 0 & -\alpha_1^{1/p} & 0 & 0 \\ \alpha_1^{-1/p} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \varrho = (0, 1, 0, 0).$$

Hence, there are two orbits with representatives $(0, 0, 0, 0)$ and $(0, 1, 0, 0)$ as claimed. The corresponding Lie algebras are

$$\begin{aligned} L_{4,2}^5 &= \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2] = x_3, x_3^{[p]} = x_4 \rangle; \\ L_{4,2}^6 &= \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2] = x_3, x_2^{[p]} = x_3, x_3^{[p]} = x_4 \rangle. \end{aligned}$$

5.2.2. *Characteristic 2.* Suppose now that $\text{char } \mathbb{F} = 2$. Suppose that \mathbb{K} and ξ are as in Section 4.2. We claim that L is isomorphic to

$$K_{4,2}^4(\xi) = \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2] = x_3, x_3^{[2]} = x_4, x_1^{[2]} = x_3, x_2^{[2]} = \xi x_3 \rangle,$$

where $\xi \in \mathbb{F}$. Further, $K_{4,2}^4(\xi_1) \cong K_{4,2}^4(\xi_2)$ if and only if $\xi_1 + \xi_2 \in \mathbb{K}$ as in Section 4.2. If a $[2]$ -map is represented by a vector $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ as above and $A \in \text{Aut}(L)$, then $A\varphi A^{-1}$ is represented by $(\alpha'_1, \alpha'_2, \beta'_1, \beta'_2)$ where

$$\begin{aligned} \alpha'_1 &= d^{-1}(\alpha_1 a_{11}^p + \alpha_2 a_{12}^p + a_{11} a_{12}) \\ \alpha'_2 &= d^{-1}(\alpha_1 a_{21}^p + \alpha_2 a_{22}^p + a_{21} a_{22}) \\ \beta'_1 &= a_{44}^{-1}(\beta_1 a_{11}^p + \beta_2 a_{12}^p + a_{13}^p) \\ \beta'_2 &= a_{44}^{-1}(\beta_1 a_{21}^p + \beta_2 a_{22}^p + a_{23}^p). \end{aligned}$$

Note that the quotient $L/\langle x_4 \rangle$ is isomorphic to the Heisenberg Lie algebra and so by Section 4.2, we may assume that $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ is of the form $(1, \xi, \beta_1, \beta_2)$. Then we use similar arguments as in the case of $\text{char } \mathbb{F} \geq 3$ to show that $(1, \xi, \beta_1, \beta_2)$ and $(1, \xi, 0, 0)$ are in the same orbit. This shows that our algebra is isomorphic to $K_{4,2}^4(\xi)$ with some $\xi \in \mathbb{F}$. To prove the claim concerning the isomorphisms among these algebras, notice that $\langle x_4 \rangle$ is the unique restricted ideal of $K_{4,2}^4(\xi)$ with dimension 1, and $K_{4,2}^4(\xi)/\langle x_4 \rangle \cong K_{3,2}^1(\xi)$. Thus $K_{4,2}^4(\xi_1) \cong K_{4,2}^4(\xi_2)$ if and only if $K_{3,2}^1(\xi_1) \cong K_{3,2}^1(\xi_2)$ if and only if $\xi_1 + \xi_2 \in \mathbb{K}$.

5.3. Finally, we consider the case where $x_3^{[p]} = 0$ and $x_4^{[p]} = x_3$. Let φ be a semilinear map that extends this $[p]$ -map to the whole L . Note that if A is an automorphism of L then A preserves $\varphi|_{Z(L)}$ if and only if $d = a_{44}^p$. Let $x_1\varphi = \alpha_1 x_3 + \beta_1 x_4$ and $x_2\varphi = \alpha_2 x_3 + \beta_2 x_4$.

5.3.1. *Odd characteristic.* Let $A \in \text{Aut}(L)$ and consider the semilinear transformation $\varphi' = A\varphi A^{-1}$ given by $x_1\varphi' = \alpha'_1 x_3 + \beta'_1 x_4$ and $x_2\varphi' = \alpha'_2 x_3 + \beta'_2 x_4$. Then

$$\begin{aligned}\alpha'_1 &= d^{-1}(\alpha_1 a_{11}^p + \alpha_2 a_{12}^p + a_{14}^p) - a_{43}/(da_{44})(\beta_1 a_{11}^p + \beta_2 a_{12}^p) \\ \alpha'_2 &= d^{-1}(\alpha_1 a_{21}^p + \alpha_2 a_{22}^p + a_{24}^p) - a_{43}/(da_{44})(\beta_1 a_{21}^p + \beta_2 a_{22}^p) \\ \beta'_1 &= a_{44}^{-1}(\beta_1 a_{11}^p + \beta_2 a_{12}^p) \\ \beta'_2 &= a_{44}^{-1}(\beta_1 a_{21}^p + \beta_2 a_{22}^p).\end{aligned}$$

As usual, we denote this action of $\text{Aut}(L)$ by ϱ . As in Section 5.2, the vectors $(\alpha_1, \beta_1, \alpha_2, \beta_2)$ and $(\alpha_3, \beta_1, \alpha_4, \beta_2)$ are in the same orbit for all α_i, β_i . Hence elements of the form $(\alpha_1, 0, \alpha_2, 0)$ form a single orbit with representative $(0, 0, 0, 0)$. Suppose now that $(\beta_1, \beta_2) \neq (0, 0)$. Consider the vector $(\alpha_1, \beta_1, \alpha_2, \beta_2)$. We may assume without loss of generality that $\alpha_1 = \alpha_2 = 0$. Then if $\beta_2 \neq 0$ then

$$(0, \beta_1, 0, \beta_2) \begin{pmatrix} \beta_2^{1/p} & -\beta_1^{1/p} & 0 & 0 \\ 0 & \beta_2^{-1/p} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \varrho = (0, 0, 0, 1).$$

If $\beta_2 = 0$ then

$$(0, \beta_1, 0, 0) \begin{pmatrix} 0 & -\beta_1^{1/p} & 0 & 0 \\ \beta_1^{-1/p} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \varrho = (0, 0, 0, 1).$$

Hence, there are two orbits with representatives $(0, 0, 0, 0)$ and $(0, 0, 1, 0)$, as claimed. The corresponding Lie algebras are

$$\begin{aligned}L_{4,2}^7 &= \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2] = x_3, x_4^{[p]} = x_3 \rangle; \\ L_{4,2}^8 &= \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2] = x_3, x_4^{[p]} = x_3, x_2^{[p]} = x_4 \rangle.\end{aligned}$$

5.3.2. *Characteristic 2.* Then $A\varphi A^{-1}$ is represented by $(\alpha'_1, \beta'_1, \alpha'_2, \beta'_2)$ where

$$\begin{aligned}\alpha'_1 &= d^{-1}(\alpha_1 a_{11}^2 + \alpha_2 a_{12}^2 + a_{14}^2 + a_{11} a_{12}) - a_{43}/(da_{44})(\beta_1 a_{11}^2 + \beta_2 a_{12}^2) \\ \alpha'_2 &= d^{-1}(\alpha_1 a_{21}^2 + \alpha_2 a_{22}^2 + a_{24}^2 + a_{21} a_{22}) - a_{43}/(da_{44})(\beta_1 a_{21}^2 + \beta_2 a_{22}^2) \\ \beta'_1 &= a_{44}^{-1}(\beta_1 a_{11}^2 + \beta_2 a_{12}^2) \\ \beta'_2 &= a_{44}^{-1}(\beta_1 a_{21}^2 + \beta_2 a_{22}^2).\end{aligned}$$

We claim that L is isomorphic to one of the following algebras:

$$\begin{aligned}K_{4,2}^5 &= \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2] = x_3, x_4^{[2]} = x_3 \rangle; \\ K_{4,2}^6 &= \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2] = x_3, x_4^{[2]} = x_3, x_2^{[2]} = x_4 \rangle.\end{aligned}$$

Similarly as in the case of $\text{char } \mathbb{F} \geq 3$ we obtain that every $\text{Aut}(L)$ -orbit contains a tuple of the form $(0, \beta_1, 0, \beta_2)$. Hence we may assume that $x_1^{[2]} = \beta_1 x_4$ and $x_2^{[2]} = \beta_2 x_4$. If $\beta_1 = \beta_2 = 0$ then L is isomorphic to $K_{4,2}^5$. If $\beta_1 = 0$ and $\beta_2 \neq 0$, then apply the diagonal automorphism $\text{diag}(\beta_2^{1/2}, \beta_2^{-1/2}, 1, 1)$ to obtain $L_{4,2}^6$. If $\beta_1 \neq 0$ and $\beta_2 \neq 0$, then we can swap x_1 and x_2 and repeat the argument. Finally if $\beta_1 \beta_2 \neq 0$, then apply the diagonal automorphism $\text{diag}(\beta_1^{-3/4} \beta_2^{-1/4}, \beta_1^{-1/4} \beta_2^{-3/4}, \beta_1^{-1} \beta_2^{-1}, \beta_1^{-1/2} \beta_2^{-1/2}) /$ to obtain that $\beta_1 = \beta_2 = 1$. Now replace x_1 with $x_1 + x_2 + x_4$ to obtain $K_{4,2}^6$. As $(K_{4,2}^5)^{[2]} \leq (K_{4,2}^5)'$, while this is not the case with $K_{4,2}^6$, the two algebras are non-isomorphic.

6. RESTRICTION MAPS ON $L_{4,3}$

Consider the Lie algebra

$$L = L_{4,3} = \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4 \rangle.$$

The automorphism group of L , with respect to the given basis, consists of the invertible matrices of the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & d_1 & a_{11}a_{23} \\ 0 & 0 & 0 & a_{11}d_1 \end{pmatrix},$$

where $d_1 = a_{11}a_{22}$. We have $Z(L) = \langle x_4 \rangle$. Since L is $[p]$ -nilpotent, we must have $x_4^{[p]} = 0$. First we note that $L_{4,3}$ is not restrictable in characteristic 2. For, a restriction map in characteristic 2 would have to satisfy $\text{ad}(x^{[2]}) = (\text{ad } x)^2$ for all x . On the other hand, we have that $(\text{ad } x_1)^2$ maps $x_1 \mapsto 0$, $x_2 \mapsto x_4$, $x_3 \mapsto 0$, $x_4 \mapsto 0$. However, this map is not an element of the algebra $\{\text{ad } x \mid x \in L_{4,3}\}$. Hence we may assume that $p \geq 3$. Then $(\text{ad } x)^p = 0$ for all x and we obtain that the codomain of the restriction map is contained in the center. Hence any $[p]$ -map is represented by a vector (α, β, γ) where

$$x_1^{[p]} = \alpha x_4, \quad x_2^{[p]} = \beta x_4, \quad x_3^{[p]} = \gamma x_4, \quad x_4^{[p]} = 0.$$

6.1. Fields of characteristic $p \geq 5$. In this case any $[p]$ -map is a semilinear transformation. Let $\varphi : L \rightarrow Z(L)$ be a semilinear transformation and $A \in \text{Aut}(L)$. Let us compute the vector $(\alpha', \beta', \gamma')$ that determines $A\varphi A^{-1}$. We have:

$$\begin{aligned} x_1 A \varphi A^{-1} &= (a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4) \varphi A^{-1} = (a_{11}^p \alpha + a_{12}^p \beta + a_{13}^p \gamma) x_4 A^{-1} \\ &= (a_{11}d_1)^{-1} (a_{11}^p \alpha + a_{12}^p \beta + a_{13}^p \gamma) x_4. \end{aligned}$$

Hence $\alpha' = (a_{11}d_1)^{-1} (a_{11}^p \alpha + a_{12}^p \beta + a_{13}^p \gamma)$. We obtain similarly, that $\beta' = (a_{11}d_1)^{-1} (a_{22}^p \beta + a_{23}^p \gamma)$ and $\gamma' = (a_{11}d_1)^{-1} d_1^p$. As above, we let ϱ denote this action of $\text{Aut}(L)$ on \mathbb{F}^3 . We need to determine the orbits of vectors $(\alpha, \beta, \gamma) \in \mathbb{F}^3$ under the action ϱ .

Let $v = (\alpha, \beta, \gamma) \in \mathbb{F}^3$. If $v = (0, 0, 0)$, then $\{v\}$ is clearly an $\text{Aut}(L)$ -orbit. Suppose that $\gamma \neq 0$. Then

$$(\alpha, \beta, \gamma) \begin{pmatrix} \gamma^{1/p} & 0 & -\alpha^{1/p} & 0 \\ 0 & \gamma^{1/p} & -\beta^{1/p} & 0 \\ 0 & 0 & \gamma^{2/p} & -\gamma^{1/p}\beta^{1/p} \\ 0 & 0 & 0 & \gamma^{3/p} \end{pmatrix} \varrho = (0, 0, \gamma_1)$$

with $\gamma_1 \in \mathbb{F}$. Next, if $\gamma \in \mathbb{F} \setminus 0$, then $(0, 0, \gamma) = (0, 0, 1) \text{diag}(\gamma^{-1/p}, \gamma^{2/p}, \gamma^{1/p}, 1) \varrho$. Hence the set of vectors (α, β, γ) with $\gamma \neq 0$ form an single orbit with orbit representative $(0, 0, 1)$.

Next, if $\gamma = 0$, but $\beta \neq 0$, then

$$(\alpha, \beta, 0) \begin{pmatrix} 1 & -(\alpha/\beta)^{1/p} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \varrho = (0, \beta, 0)$$

with $\beta_1 \in \mathbb{F}$.

We claim that $(0, \beta_1, 0)$ and $(0, \beta_2, 0)$ are in the same $\text{Aut}(L)$ -orbits if and only if $\beta_1\beta_2^{-1}$ is a square in \mathbb{F} . First assume that $(0, \beta_1, 0)$ and $(0, \beta_2, 0)$ are in the same $\text{Aut}(L)$ -orbits. Then

$$(0, \beta_2, 0) = (0, \beta_1, 0)(A\varrho) = (a_{11}d_1)^{-1}(\beta_1a_{12}^p, \beta_1a_{22}^p, 0).$$

From this we obtain that $a_{12} = 0$ and that $\beta_2 = (a_{11}d_1)^{-1}a_{22}^p\beta_1$ which gives that $\beta_1\beta_2^{-1} = a_{11}^2a_{22}^{1-p}$. Since p is odd, $\beta_1\beta_2^{-1} \in (\mathbb{F}^*)^2$. Assume next that $\beta_1\beta_2^{-1} = \varepsilon^2$ with some $\varepsilon \in \mathbb{F}$. Then

$$(0, \beta_1, 0) \text{diag}(\varepsilon, 1, \varepsilon, \varepsilon^2) \varrho = (0, \varepsilon^{-2}\beta_1, 0) = (0, \beta_2, 0).$$

Hence two elements $(0, \beta_1, 0)$ and $(0, \beta_2, 0)$ are in the same orbit if and only if $\beta_1\beta_2^{-1}$ is a square.

Finally, if $\gamma = \beta = 0$, but $\alpha \neq 0$, then $(\alpha, 0, 0) \text{diag}(1, \alpha, \alpha, \alpha) \varrho = (1, 0, 0)$.

Thus we obtain the following elements are $\text{Aut}(L)$ -orbit representatives: $(0, 0, 0)$, $(1, 0, 0)$, $(0, \beta, 0)$, $(0, 0, 1)$. Further $(0, \beta_1, 0)$ and $(0, \beta_2, 0)$ are in the same orbit if and only if $\beta_1\beta_2^{-1}$ is a square.

Hence, up to isomorphism, the possible restricted Lie algebra structures on L are as follows:

$$\begin{aligned} L_{4,3}^1 &= \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4 \rangle; \\ L_{4,3}^2 &= \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_1^{[p]} = x_4 \rangle; \\ L_{4,3}^3(\beta) &= \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_2^{[p]} = \beta x_4 \rangle; \\ L_{4,3}^4 &= \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_3^{[p]} = x_4 \rangle \end{aligned}$$

where $\beta \in \mathbb{F}^*$. Further $L_{4,1}^3(\beta_1) \cong L_{4,1}^3(\beta_2)$ if and only if $\beta_1\beta_2^{-1}$ is a square.

6.2. Fields of characteristic 3. In this case we have

$$(ax_1 + bx_2 + cx_3 + dx_4)^{[3]} = a^3x_1^{[3]} + b^3x_2^{[3]} + c^3x_3^{[3]} + a^2bx_4.$$

If φ is a $[3]$ -map then, as in the general case, φ is represented by a vector (α, β, γ) where

$$x_1\varphi = \alpha x_4, \quad x_2\varphi = \beta x_4, \quad x_3\varphi = \gamma x_4, \quad x_4\varphi = 0.$$

Let us compute the vector $(\alpha', \beta', \gamma')$ that determines $A\varphi A^{-1}$ with $A \in \text{Aut}(L)$:

$$\begin{aligned} x_1 A\varphi A^{-1} &= (a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4)\varphi A^{-1} \\ &= (a_{11}^3\alpha + a_{12}^3\beta + a_{13}^3\gamma + a_{11}^2a_{12})x_4 A^{-1} \\ &= (a_{11}d_1)^{-1}(a_{11}^3\alpha + a_{12}^3\beta + a_{13}^3\gamma + a_{11}^2a_{12})x_4. \end{aligned}$$

Hence $\alpha' = (a_{11}d_1)^{-1}(a_{11}^3\alpha + a_{12}^3\beta + a_{13}^3\gamma + a_{11}^2a_{12})$ and also $\beta' = (a_{11}d_1)^{-1}(a_{22}^3\beta + a_{23}^3\gamma)$ and $\gamma' = (a_{11}d_1)^{-1}d_1^3 = a_{11}^{-1}d_1^2$. Thus in matrix form we obtain that the right action ϱ on the set of $[3]$ -maps can be written as

$$(8) \quad (\alpha', \beta', \gamma') = (\alpha, \beta, \gamma)A\varrho = (a_{11}^2a_{22})^{-1}(\alpha, \beta, \gamma) \begin{pmatrix} a_{11}^3 & 0 & 0 \\ a_{12}^3 & a_{22}^3 & 0 \\ a_{13}^3 & a_{23}^3 & d_1^3 \end{pmatrix} + (a_{22}^{-1}a_{12}, 0, 0).$$

Note that ϱ is not a linear action. Let $v = (\alpha, \beta, \gamma) \in \mathbb{F}^3$. Suppose that $\gamma \neq 0$. Then

$$(\alpha, \beta, \gamma) \begin{pmatrix} \gamma^{1/3} & 0 & -\alpha^{1/3} & 0 \\ 0 & \gamma^{1/3} & -\beta^{1/3} & 0 \\ 0 & 0 & \gamma^{2/3} & -\gamma^{1/3}\beta^{1/3} \\ 0 & 0 & 0 & \gamma \end{pmatrix} \varrho = (0, 0, \gamma^2),$$

and

$$(0, 0, \gamma^2) \text{diag}(\gamma^{-2/3}, \gamma^{-2/3}, \gamma^{-4/3}, \gamma^{-2}) \varrho = (0, 0, 1).$$

Hence the vectors (α, β, γ) , with $\gamma \neq 0$, represent the same restricted Lie algebra as $(0, 0, 1)$.

We notice that if $A \in \text{Aut}(L)$ as above then $(0, 0, 0)A\varrho = (a_{12}a_{22}^{-1}, 0, 0)$ and hence the $[p]$ -maps represented by the vectors of the form $(\alpha, 0, 0)$ form a single orbit with orbit representative $(0, 0, 0)$.

It remains to describe the orbits of the $[p]$ -maps that are represented by the vectors of the form $(\alpha, \beta, 0)$ with $\beta \neq 0$. For $\beta \in \mathbb{F}^*$, let \mathbb{K}_β denote the set $\{\beta x^3 + x \mid x \in \mathbb{F}\}$. Then \mathbb{K}_β is an \mathbb{F}_3 -subspace, but it depends on \mathbb{F} and on β . For instance if \mathbb{F} is finite and $\beta x^3 + x$ has a solution other than 0, then it has codimension 1, otherwise $\mathbb{K}_\beta = \mathbb{F}$.

Lemma 6.1. *Let $\alpha_1, \alpha_2 \in \mathbb{F}$ and $\beta_1, \beta_2 \in \mathbb{F}^*$. Then $(\alpha_1, \beta_1, 0)$ and $(\alpha_2, \beta_2, 0)$ represent isomorphic restricted Lie algebras if and only if $\beta_1\beta_2^{-1}$ is a square and $\alpha_2\sqrt{\beta_2/\beta_1} + \alpha_1 \in \mathbb{K}_{\beta_1}$ or $\alpha_2\sqrt{\beta_2/\beta_1} - \alpha_1 \in \mathbb{K}_{\beta_1}$.*

Proof. The vectors $(\alpha_1, \beta_1, 0)$ and $(\alpha_2, \beta_2, 0)$ represent isomorphic restricted Lie algebras if and only if there exists $A \in \text{Aut}(L)$ such that $(\alpha_1, \beta_1, 0)A\varrho = (\alpha_2, \beta_2, 0)$. That is

$$(9) \quad (\alpha_2, \beta_2, 0) = (\alpha_1, \beta_1, 0)A\varrho = a_{11}^{-2}a_{22}^{-1}(a_{11}^3\alpha_1 + a_{12}^3\beta_1 + a_{11}^2a_{12}, a_{22}^3\beta_1, 0).$$

Now we deduce that $\beta_2 = a_{11}^{-2}a_{22}^2\beta_1$. In particular β_2/β_1 is a square. Furthermore, as $a_{11}a_{22}^{-1} = \pm\sqrt{\beta_1/\beta_2}$ (meaning that the equality holds with plus or minus),

$$\begin{aligned} \alpha_2 &= \pm a_{11}^{-3}\sqrt{\beta_1/\beta_2}(a_{11}^3\alpha_1 + a_{12}^3\beta_1 + a_{11}^2a_{12}) = \\ &\quad \pm (\sqrt{\beta_1/\beta_2}\alpha_1 + \sqrt{\beta_1/\beta_2}(a_{11}^{-1}a_{12})^3\beta_1 + \sqrt{\beta_1/\beta_2}a_{11}^{-1}a_{12}). \end{aligned}$$

Thus

$$\sqrt{\beta_2/\beta_1}\alpha_2 = \pm(\alpha_1 + (a_{11}^{-1}a_{12})^3\beta_1 + a_{11}^{-1}a_{12}).$$

Therefore $\sqrt{\beta_2/\beta_1}\alpha_2 \pm \alpha_1 \in \mathbb{K}_{\beta_1}$, as claimed.

Conversely, suppose that $\alpha_2\sqrt{\beta_2/\beta_1} + \alpha_1 \in \mathbb{K}_{\beta_1}$ or $\alpha_2\sqrt{\beta_2/\beta_1} - \alpha_1 \in \mathbb{K}_{\beta_1}$. Choose $\delta \in \mathbb{F}$ such that, in the latter case, $\sqrt{\beta_2/\beta_1}\alpha_2 - \alpha_1 = \beta_1\delta^3 + \delta$, while $-\sqrt{\beta_2/\beta_1}\alpha_2 - \alpha_1 = \beta_1\delta^3 + \delta$ in the former. Then apply the automorphism

$$A = \begin{pmatrix} 1 & \delta & 0 & 0 \\ 0 & \mp\sqrt{\beta_2/\beta_1} & 0 & 0 \\ 0 & 0 & \mp\sqrt{\beta_2/\beta_1} & 0 \\ 0 & 0 & 0 & \mp\sqrt{\beta_2/\beta_1} \end{pmatrix}$$

with the sign of $\sqrt{\beta_2/\beta_1}$ chosen accordingly. It is straight to see that $(\alpha_1, \beta_1, 0)A\varrho = (\alpha_2, \beta_2, 0)$. \square

Hence, up to isomorphism, we have the following restricted Lie algebras with the underlying Lie algebra $L_{4,3}$ over a field of characteristic three:

$$\begin{aligned} K_{4,3}^1 &= \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4 \rangle; \\ K_{4,3}^2 &= \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_3^{[3]} = x_4 \rangle; \\ K_{4,3}^3(\alpha, \beta) &= \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, x_1^{[3]} = \alpha x_4, x_2^{[3]} = \beta x_4 \rangle \end{aligned}$$

where $\alpha \in \mathbb{F}$ and $\beta \in \mathbb{F}^*$. Furthermore, $K_{4,3}^2(\alpha_1, \beta_1) \cong K_{4,3}^2(\alpha_2, \beta_2)$ if and only if $\beta_1\beta_2^{-1}$ is a square and $\alpha_2\sqrt{\beta_2/\beta_1} + \alpha_1 \in \mathbb{K}_{\beta_1}$ or $\alpha_2\sqrt{\beta_2/\beta_1} - \alpha_1 \in \mathbb{K}_{\beta_1}$. The arguments in the section show that these algebras are pairwise non-isomorphic.

6.3. Characteristic 2. As noted at the beginning of the section, in characteristic 2 the Lie algebra is not restrictable.

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